Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 5, pp. 902–911, 2001

STRESSES IN AN ELASTIC BODY UNDER NONLINEAR ANTIPLANE DEFORMATION

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UDC 539.3

The stress field in a cylindrical elastic body under antiplane deformation and certain constraints imposed on volume and surface forces is studied in a nonlinear formulation in actual-state variables. A boundary-value problem for independent stress components is formulated in Cartesian and complex variables, sufficient ellipticity conditions for this problem are indicated, and constraints on surface loading are imposed. Analytical solutions are given for linear and weak nonlinear elastic potentials. Similarity to a plane subsonic ideal-gas flow is established. An approximate method for the solution of the problem is developed.

In the present paper, using the nonlinear theory of elasticity in actual-state variables, we consider the stressed state of an elastic cylindrical body with a specified elastic potential under longitudinal antiplane deformation without volume forces and with constant surface loading along the cylinder generatrix. Stresses can be determined from the equations of equilibrium and stress compatibility in the volume of the body and from force conditions on its surface.

In the actual-state Cartesian coordinate system x_1, x_2, x_3 with the x_3 axis parallel to the cylinder generatrix (x_3 is a longitudinal coordinate) and the $x_1 = x$ and $x_2 = y$ axes in the plane of its middle cross section S with boundary L (x and y are transverse coordinates), antiplane strain is described by the displacements $u_1 = u_2 = 0$ and $u_3 = w(x, y)$. In these variables, the strain measure is the Almansi tensor. In the case of antiplane deformation, the components E_{kl} and invariants E_k of the tensor given by the formulas [1]

$$2E_{kl} = \partial_k u_l + \partial_l u_k - \partial_k u_m \partial_l u_m,$$

$$E_1 = E_{mm}, \qquad 2E_2 = E_{mm}E_{nn} - E_{mn}E_{nm}, \qquad E_3 = \det(E_{kl})$$

(hereinafter, the subscript runs from 1 to 3, and summation is performed over repeating indices) are expressed in terms of axial displacement:

$$2E_{11} = -\Big(\frac{\partial w}{\partial x}\Big)^2, \qquad 2E_{22} = -\Big(\frac{\partial w}{\partial y}\Big)^2, \qquad 2E_{33} = 0,$$
$$2E_{12} = -\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}, \qquad 2E_{31} = \frac{\partial w}{\partial x}, \qquad 2E_{32} = \frac{\partial w}{\partial y}; \tag{1}$$

$$2E_1 = -|\nabla w|^2, \qquad 4E_2 = -|\nabla w|^2, \qquad E_3 = 0. \tag{2}$$

Hence, E_{kl} and E_k are functions of the transverse coordinates.

Generally, the equations of strain compatibility can be obtained by excluding displacements from the formulas expressing strains in terms of these variables [2]. Eliminating displacements from relations (1), we obtain the following compatibility equations for antiplane strain:

$$2E_{11} = -(2E_{31})^2, \qquad 2E_{22} = -(2E_{32})^2, \qquad 2E_{33} = 0, \tag{3}$$
$$2E_{12} = -2E_{31}2E_{32}, \qquad \frac{\partial 2E_{32}}{\partial x} - \frac{\partial 2E_{31}}{\partial y} = 0.$$

Novosibirsk State University, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 5, pp. 198–208, September–October, 2001. Original article submitted April 23, 2001.

 0021-8944/01/4205-0902 $25.00

System (3) consists of finite and differential equations. In the first four equations, the strain components are expressed in terms of two of them (E_{31} and E_{32}) by nonlinear formulas, and the last equation is a linear differential equation for the independent components. Strain invariants (2) are nonpositive, expressed in terms of the invariant E_1 ($2E_2 = E_1$ and $E_3 = 0$), and satisfy the incompressibility condition [1] $2E_1 - 4E_2 + 8E_3 = 0$. Therefore, under antiplane deformation, the material behaves as an incompressible one.

The mechanical behavior of an incompressible elastic body in actual-state variables is defined by a modified Murnaghan's law [1, 3], which relates Cauchy stresses P_{kl} with Almansi strains:

$$P_{kl} = -q^*\delta_{kl} + (\delta_{km} - 2E_{km})\frac{\partial U}{\partial E_{lm}}.$$

Here q^* is the Lagrangian factor, δ_{kl} is the Kronecker delta, and U is the elastic potential. For a homogeneous isotropic material, the elastic potential is a function of basis strain invariants. In the case considered, by virtue of the characteristics of the invariants, this potential depends only on the first invariant: $U = U(E_1)$. With allowance for this property of the potential and the relations

$$E_1 = E_{lm}\delta_{ml}, \qquad \frac{\partial E_1}{\partial E_{lm}} = \delta_{ml}, \qquad \frac{\partial U(E_1)}{\partial E_{lm}} = U'(E_1)\delta_{ml},$$

Murnaghan's law for antiplane deformation is written as the following quasilinear dependence of stresses on strains:

$$P_{kl} = -q\delta_{kl} - 2U'(E_1)E_{kl} \tag{4}$$

($q = q^* - U'$ is the hydrostatic pressure).

Inverting dependences (4), we obtain the strain relations

$$2E_{kl} = -(P_{kl} + q\delta_{kl})/U'. \tag{5}$$

The derivative of the elastic potential contained in (5) can be expressed in terms of stresses. Indeed, eliminating the invariant E_1 from the relations

$$2E_1 = 2E_{mm} = -(2E_{31})^2 - (2E_{32})^2 = -(P_{31}^2 + P_{32}^2)/U'^2, \qquad U' = T(2E_1) \tag{6}$$

[following from formulas (3) and (5)], we have the desired dependence in implicit form:

$$U' = T(-R^2/U'^2), \qquad R^2 = P_{31}^2 + P_{32}^2. \tag{7}$$

In particular, for the quadratic Rivlin–Sounders potential (coinciding with the Mooney potential in the linear case)

$$U(E_1) = aE_1^2 - 2bE_1 \qquad (a > 0, \quad b > 0, \quad E_1 < 0), \tag{8}$$

which describes large elastic strains of rubber-like materials with reasonable accuracy [4, 5], the dependence $U'(R^2)$ is given by the solution of the cubic equation

$$U'^3 + 2bU'^2 + aR^2 = 0. \tag{9}$$

The substitution $U' = V - 2b/3$ reduces this equation to an incomplete equation

$$V^3 - (4b^2/3)V + m = 0 \tag{10}$$

with the coefficients $m = aR^2 + 16b^3/27$ satisfying the inequality

$$Q = (-4b^2/9)^3 + (m/2)^2 = (8ab^3/27)R^2 + (a^2/4)R^4 > 0.$$

Equation (10) has only one real solution [6]

$$V = I_+(R^2) + I_-(R^2) \qquad (I_\pm = \sqrt[3]{-m/2 \pm \sqrt{Q}}).$$

Consequently, the solution of Eq. (9) has the form

$$U'(R^2) = I_+(R^2) + I_-(R^2) - 2b/3. \tag{11}$$

The quantities m, Q, and R^2 are determined above. Thus, strains are expressed by the inverse Murnaghan's law in terms of stress and pressure.

Substitution of strains (5) into equalities (3) yields the stress compatibility equations

$$P_{11} = -q + P_{31}^2/U', \quad P_{22} = -q + P_{32}^2/U', \quad P_{33} = -q, \quad P_{12} = P_{31}P_{32}/U',$$
$$\frac{\partial}{\partial x}\frac{P_{32}}{U'} - \frac{\partial}{\partial y}\frac{P_{31}}{U'} = 0, \tag{12}$$

where U' is determined in (11). The first four of these equations represent stresses in terms of pressure and the stresses P_{31} and P_{32}, and the last equation is a nonlinear differential equation for the independent stresses.

We assume that a specified surface (with the external normal n_m) of the cylinder is loaded with load p_k, which is represented in terms of stresses by the formula $p_k = P_{km}n_m$. On the lateral surface of the cylinder S_*, the normal components are equal to $(n_m) = (n_1(x,y), n_2(x,y), 0)$, and hence, with allowance for (12), the lateral load is written as

$$p_1 = -qn_1 + (P_{31}/U')P_{3m}n_m, \quad p_2 = -qn_2 + (P_{32}/U')P_{3m}n_m, \quad p_3 = P_{3m}n_m. \tag{13}$$

At the cylinder butt ends $S^\pm$ (the superscript plus corresponds to the upper butt end and the superscript minus corresponds to the lower butt end), the normal components are constant $[(n_m^\pm) = (0,0,\pm 1)]$ and the load is

$$p_1^\pm = \pm P_{31}, \qquad p_2^\pm = \pm P_{32}, \qquad p_3^\pm = \pm P_{33} = \mp q. \tag{14}$$

It follows from (13) and (14) that if the surface load does not depend on x_3, the pressure on the cylinder surface does not depend on x_3 as well. Next, we assume that the pressure does not depend on this coordinate over the entire volume of the cylinder: $q = q(x,y)$. Consequently, in the cylinder, the stresses (4) are functions of the transverse coordinates: $P_{kl} = P_{kl}(x,y)$.

In the absence of volume forces, taking into account the stress relations (12) and the expressions $q = q(x,y)$ and $P_{kl} = P_{kl}(x,y)$, we write the equilibrium equations $\partial P_{km}/\partial x_m = 0$ in the form

$$-\frac{\partial q}{\partial x} + \frac{P_{31}}{U'}\Big(\frac{\partial P_{31}}{\partial x} + \frac{\partial P_{32}}{\partial y}\Big) + U'\Big(\frac{P_{31}}{U'}\frac{\partial}{\partial x}\frac{P_{31}}{U'} + \frac{P_{32}}{U'}\frac{\partial}{\partial y}\frac{P_{31}}{U'}\Big) = 0,$$
$$-\frac{\partial q}{\partial y} + \frac{P_{32}}{U'}\Big(\frac{\partial P_{31}}{\partial x} + \frac{\partial P_{32}}{\partial y}\Big) + U'\Big(\frac{P_{32}}{U'}\frac{\partial}{\partial y}\frac{P_{32}}{U'} + \frac{P_{31}}{U'}\frac{\partial}{\partial x}\frac{P_{32}}{U'}\Big) = 0; \tag{15}$$

$$\frac{\partial P_{31}}{\partial x} + \frac{\partial P_{32}}{\partial y} = 0. \tag{16}$$

Equations (15) define pressure and Eq. (16) [together with the last equation in (12)] defines the independent stresses. Indeed, Eqs. (15) [with allowance for Eqs. (12) and (16)] and the notation for the sum of squares of the independent stresses (7) become

$$-\frac{\partial q}{\partial x} + U'\frac{\partial}{\partial x}\frac{R^2}{2U'^2} = 0, \qquad -\frac{\partial q}{\partial y} + U'\frac{\partial}{\partial y}\frac{R^2}{2U'^2} = 0.$$

Taking into account the first relation in (6) and equalities

$$\frac{\partial}{\partial x_k}\frac{R^2}{2U'^2} = -\frac{\partial E_1}{\partial x_k}, \qquad U'\frac{\partial}{\partial x_k}\frac{R^2}{2U'^2} = -U'\frac{\partial E_1}{\partial x_k} = -\frac{\partial U}{\partial x_k} \qquad (k = 1,\ 2),$$

we write these equations as $\partial(q+U)/\partial x = 0$ and $\partial(q+U)/\partial y = 0$. After integration, they define the hydrostatic pressure via the elastic potential:

$$q = h - U \tag{17}$$

(h is an integration constant).

The axial components $F_3^\pm$ of the resulting butt end loads (14) depend linearly on h:

$$F_3^\pm = \int_S p_3^\pm\, dS = \mp\int_S q\, dS = \mp\Big(hS - \int_S U\, dS\Big).$$

Hence, this variable can be expressed in terms of the axial load. In particular, in the absence of the axial load, the constant h is equal to an average value of the potential in the cross section of the cylinder

$$h = \frac{1}{S}\bigg(\int_S U\, dS \mp F_3^\pm\bigg) \qquad \Big(\text{for} \quad F_3^\pm = 0, \quad h = \frac{1}{S}\int_S U\, dS\Big). \tag{18}$$

In the last case, according to Eqs. (17) and (18), the hydrostatic pressure coincides with the deviation of the elastic potential from its average value.

At the boundary L of the cylinder cross section, the stresses P_{31} and P_{32} are expressed in terms of load by the nonlinear equations (13). For further consideration, we introduce linear combinations g_n and g_t of stresses (whose coefficients are the components n_1 and n_2 of the normal and the components $t_1 = -n_2$ and $t_2 = n_1$ of the tangent) that are biuniquely related to the stresses P_{31} and P_{32}:

$$g_n = P_{31}n_1 + P_{32}n_2, \qquad g_t = P_{31}t_1 + P_{32}t_2 = -P_{31}n_2 + P_{32}n_1,$$
$$P_{31} = g_n n_1 - g_t n_2, \qquad P_{32} = g_n n_2 + g_t n_1, \qquad R^2 = P_{31}^2 + P_{32}^2 = g_t^2 + g_n^2 \quad \text{on } L. \tag{19}$$

Determining g_n and g_t from (13), we then determine stresses and impose constraints on the load.

With allowance for formulas (13), (17), and (19) and the values $(t_m) = (-n_2, n_1, 0)$, $(n'_m) = (-n_1, -n_2, 0)$, and $(b_m) = (0, 0, 1)$ of the unit vectors of the natural contour axes, the natural components p_t, $p_{n'}$, and p_b (tangent, normal, and binormal) of the contour load vector are equal to

$$p_t = p_m t_m = g_n g_t/U', \quad -p_{n'} = p_n = p_m n_m = U - h + g_n^2/U', \quad p_b = p_m b_m = g_n. \tag{20}$$

In (20), the last equality defines the quantity g_n:

$$g_n = p_b; \tag{21}$$

the first two equalities define the quantity g_t. Indeed, substituting the quantity $R^2 = g_n^2 + g_t^2$ into Eq. (9) considered at the boundary and the elastic potential (8) expressed in terms of its derivative $U = (U'^2 - 4b^2)/(4a)$ into the second equation in (20), we obtain the following equations for U':

$$U'^3 + 2bU'^2 + a(g_n^2 + g_t^2) = 0, \quad U'^3 - 4BU' + 4ag_n^2 = 0 \qquad [B = b^2 + a(h + p_n)].$$

Substituting the expressions $g_n = p_b$ and $U' = p_b g_t/p_t$ from the first and third equalities in (20) into the above equations, we obtain the cubic equations for g_t:

$$p_b^3 g_t^3 + Ap_t g_t^2 + ap_b^2 p_t^3 = 0, \quad p_b^2 g_t^3 - 4Bp_t^2 g_t + 4ap_t^3 p_b = 0 \qquad (A = 2bp_b^2 + ap_t^2).$$

The first equation has the following unique real solution for arbitrary parameters [6]:

$$g_t = \frac{p_t}{p_b}\Big(\sqrt[3]{-\frac{d}{2} + \sqrt{M}} + \sqrt[3]{-\frac{d}{2} - \sqrt{M}} - \frac{A}{3p_b^2}\Big) \quad \Big(d = ap_b^2 + \frac{2A^3}{27p_b^6} \quad M = \frac{a^2p_b^4}{4} + \frac{aA^3}{27p_b^4}\Big); \tag{22}$$

the second equation has the unique real solution

$$g_t = \frac{p_t}{p_b}\Big(\sqrt[3]{-2ap_b^2 + 2\sqrt{N}} + \sqrt[3]{-2ap_b^2 - 2\sqrt{N}}\Big) \tag{23}$$

for the parameters satisfying the inequality

$$N = a^2p_b^4 - \frac{16}{27}B^3 > 0.$$

This inequality (23) and the condition matching Eqs. (22) and (23) for the quantity g_t

$$\sqrt[3]{-d/2 + \sqrt{M}} + \sqrt[3]{-d/2 - \sqrt{M}} - A/(3p_b^2) = \sqrt[3]{-2ap_b^2 + 2\sqrt{N}} + \sqrt[3]{-2ap_b^2 - 2\sqrt{N}}$$

impose constraints on the surface load that ensure antiplane deformation of the cylinder. Thus, the boundary values of the independent stresses are given by formulas (19) in which the quantities g_t and g_n are defined by Eqs. (21) and (22) for a specified load satisfying the constraints.

The differential equations (12) and (16) [U' is expressed in terms of stresses by formula (11)] and boundary conditions (19) define the following nonlinear boundary-value problem for the independent Cartesian stresses:

$$\frac{\partial}{\partial x}\frac{P_{32}}{U'} - \frac{\partial}{\partial y}\frac{P_{31}}{U'} = 0, \quad \frac{\partial P_{31}}{\partial x} + \frac{\partial P_{32}}{\partial y} = 0,$$
$$P_{31} = g_n n_1 - g_t n_2, \quad P_{32} = g_n n_2 + g_t n_1 \qquad \text{on} \quad L. \tag{24}$$

Let us write the equations in expanded form and establish the ellipticity conditions for this system.

If we differentiate the derivative $U'(E_1)$ with respect to the coordinates and allow for the expression of the invariant $E_1 = -R^2/(2U'^2)$, the gradients $\partial U'/\partial x$ and $\partial U'/\partial y$ are written as

$$\frac{\partial U'}{\partial x} = \frac{U'U''}{2(R^2U'' - U'^3)}\,\frac{\partial}{\partial x}(P_{31}^2 + P_{32}^2), \qquad \frac{\partial U'}{\partial y} = \frac{U'U''}{2(R^2U'' - U'^3)}\,\frac{\partial}{\partial y}(P_{31}^2 + P_{32}^2).$$

With allowance for the above relations, Eqs. (24) become

$$H_1 = U'P_{31}P_{32}\Big(\frac{\partial P_{31}}{\partial x} - \frac{\partial P_{32}}{\partial y}\Big) + (U''P_{32}^2 - U'^3)\frac{\partial P_{31}}{\partial y} - (U''P_{31}^2 - U'^3)\frac{\partial P_{32}}{\partial x} = 0,$$
$$H_2 = \frac{\partial P_{31}}{\partial x} + \frac{\partial P_{32}}{\partial y} = 0. \tag{25}$$

Let us consider the characteristic determinant D of this system [7]. Denoting the desired quantities by $w_1 = P_{31}$ and $w_2 = P_{32}$, we write the determinant in the form

$$D = \det(A_{kl}), \qquad A_{kl} = \frac{\partial H_k}{\partial(\partial w_l/\partial x_m)}\,v_m, \qquad A_{21} = v_1, \quad A_{22} = v_2,$$

$$A_{11} = U''w_1w_2v_1 + (U''w_2^2 - U'^3)v_2, \qquad A_{12} = (-U''w_1^2 + U'^3)v_1 - U''w_1w_2v_2.$$

The value of this determinant is

$$D = A_{11}A_{22} - A_{21}A_{12} = -U'^3(v_1^2 + v_2^2) + U''(w_1v_1 + w_2v_2)^2.$$

From this it follows that

$$D > 0 \quad \text{for} \quad U' < 0, \quad U'' \geqslant 0, \qquad D < 0 \quad \text{for} \quad U' > 0, \quad U'' \leqslant 0. \tag{26}$$

If the elastic potential satisfies conditions (26) (the first or the second condition), the characteristic equation $D = 0$ has no real roots. Therefore, the nonlinear system (25) is elliptic for any solution. Thus, inequalities (26) are sufficient conditions for ellipticity of the equations of antiplane deformation of an elastic material.

For the quadratic Rivlin–Sounders elastic potential (8), the derivatives have the values $U' = 2aE_1 - 2b < 0$ and $U'' = 2a > 0$ ($a > 0$, $b > 0$, and $E_1 < 0$); therefore, for this potential, the ellipticity conditions (26) are satisfied.

The boundary-value problem (24) can be written in complex form. Let us convert from the Cartesian coordinates x_m to the complex coordinates z^k:

$$z^1 = z = x + iy, \quad z^2 = \bar{z} = x - iy, \quad z^3 = x_3,$$

$$2\,\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}, \qquad 2\,\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z^3} = \frac{\partial}{\partial x_3}.$$

The complex components of the normal (n^k) and the stresses (P^{kl}) are expressed in terms of the Cartesian components of the corresponding quantities by the transformation formulas $n^k = n_m\,\partial z^k/\partial x_m$ and $P^{kl} = P_{ms}\,(\partial z^k/\partial x_m)(\partial z^l/\partial x_s)$ in the following form [8–10]:

$$n^1 = \bar{n}^2 = n_1 + in_2, \qquad n^3 = n_3, \qquad P^{11} = \bar{P}^{22} = P_{11} - P_{22} + 2iP_{12},$$
$$P^{12} = P_{11} + P_{22}, \qquad P^{31} = \bar{P}^{32} = P_{31} + iP_{32}, \qquad P^{33} = P_{33}. \tag{27}$$

At the points of the contour L near the normal to the lateral surface of the cylinder, the third component is zero, and the first and second components are expressed in terms of the Cartesian and complex equations of the contour L [$x = x(s)$, $y = y(s)$ and $z = z(s)$, $\bar{z} = \bar{z}(s)$ (s is an arc of the contour)] by the formulas $n_1 = dy/ds$ and $n_2 = -dx/ds$; $n^1 = \bar{n}^2 = -i\,dz/ds$. By virtue of (7), (12), and (17), the complex stresses (27) are expressed in terms of the component P^{31}:

$$P^{11} = \bar{P}^{22} = (P^{31})^2/U'(R^2), \quad P^{12} = 2(U(R^2) - h) + R^2/U'(R^2),$$
$$P^{33} = U(R^2) - h, \quad P^{32} = \bar{P}^{31}, \quad R^2 = P^{31}\bar{P}^{31}. \tag{28}$$

In the complex variables, the nonlinear boundary-value problem (24) for stresses has the form

$$\frac{\partial}{\partial z}\,\frac{P^{31}}{U'} - \frac{\partial}{\partial \bar{z}}\,\frac{\bar{P}^{31}}{U'} = 0, \qquad \frac{\partial P^{31}}{\partial z} + \frac{\partial \bar{P}^{31}}{\partial \bar{z}} = 0, \qquad P^{31}\,\frac{d\bar{z}}{ds}\Big|_L = g_t - ig_n, \tag{29}$$

where the dependence $U' = U'(R^2)$ ($R^2 = P^{31}\bar{P}^{31}$) is given by (11).

In the case of a linear elastic potential [$a = 0$ in (8)], its derivative is constant: $U' = -2b = \text{const}$. By virtue of this property, problem (29) becomes linear (i.e., coinciding with the corresponding problem of linear elasticity):

$$\frac{\partial P^{31}}{\partial z} - \frac{\partial \bar{P}^{31}}{\partial \bar{z}} = 0, \quad \frac{\partial P^{31}}{\partial z} + \frac{\partial \bar{P}^{31}}{\partial \bar{z}} = 0, \quad P^{31} \frac{d\bar{z}}{ds}\Big|_L = g_t - i g_n.$$

A consequence of these equations is the equation $\partial P^{31}/\partial z = 0$. After integration, it defines the complex stress in terms of an arbitrary function $\bar{\varphi}'(\bar{z})$ (complex potential). Substitution of the stress relation into the boundary condition yields the following boundary-value problem for the potential:

$$P^{31} = \bar{\varphi}'(\bar{z}), \quad \varphi(z)\Big|_L = g(s) + G, \quad g(s) = \int_0^s (g_t + i g_n)\, ds, \quad G = \text{const}. \tag{30}$$

If S is a simply connected region (finite or infinite) bounded by a simple smooth contour L in the plane z, it can be conformally mapped onto a unit circle K with circumference C in a plane ζ by means of a holomorphic function $z = \omega(\zeta)$ [$\omega'(\zeta) \neq 0$]. Upon mapping, the complex potential and its derivative take values of $\varphi(z) = \varphi(\zeta)$ and $\varphi'(z) = \varphi'(\zeta)/\omega'(\zeta)$, which, according to (30), allows us to express the stress in terms of both the potential and the mapping function and write the following boundary-value problem for the transformed potential at the boundary of the unit circle (without loss of generality assuming that $G = 0$):

$$P^{31} = \bar{\varphi}'(\bar{\zeta})/\bar{\omega}'(\bar{\zeta}), \quad \varphi(\sigma) = g(\sigma), \quad \zeta = r e^{i\theta} \in K, \quad \sigma = e^{i\theta} \in C. \tag{31}$$

For the quadratic elastic potential (8) in the case of weak nonlinearity [where the coefficient of the quadratic term is small compared to the coefficient of the linear term, i.e., $c = a/(2b) \ll 1$], one obtains an approximate analytical solution of problem (29). In a linear approximation of the small parameter, the quantities considered can be written as

$$\begin{gathered} P^{31} = P_0^{31} + c P_1^{31}, \quad U' = U_0' + c U_1', \quad g_t = g_{t0} + c g_{t1}, \quad g_n = g_{n0} + c g_{n1}, \\ R^2 = (R^2)_0 + c (R^2)_1, \quad U = U_0 + c U_1, \end{gathered} \tag{32}$$

where, with allowance for (9), (21), (22), and (28), we have

$$\begin{gathered} U_0' = -2b, \quad U_1' = -(R^2)_0/(2b), \quad (R^2)_0 = P_0^{31} \bar{P}_0^{31}, \quad (R^2)_1 = P_0^{31} \bar{P}_1^{31} + P_1^{31} \bar{P}_0^{31}, \\ g_{t0} = -2b p_t/p_b, \quad g_{t1} = -p_t (4b^2 p_t^2 + p_b^4)/(2b p_b^3), \quad g_{n0} = p_b, \quad g_{n1} = 0, \\ U_0 = (R^2)_0/(4b), \quad U_1 = (R^2)_0/(32 b^3). \end{gathered} \tag{33}$$

Substituting quantities (32) into relations (29) and equating the coefficients at equal powers of the parameter on both sides, we obtain linear boundary-value problems for the zero and first stress components:

$$\frac{\partial P_0^{31}}{\partial z} - \frac{\partial \bar{P}_0^{31}}{\partial \bar{z}} = 0, \quad \frac{\partial P_0^{31}}{\partial z} + \frac{\partial \bar{P}_0^{31}}{\partial \bar{z}} = 0, \quad P_0^{31} \frac{d\bar{z}}{ds}\Big|_L = g_{t0} - i g_{n0}; \tag{34}$$

$$\begin{gathered} \frac{\partial P_1^{31}}{\partial z} - \frac{\partial \bar{P}_1^{31}}{\partial \bar{z}} + \frac{3}{4b^2} \Big[(\bar{P}_0^{31})^2 \frac{\partial P_0^{31}}{\partial \bar{z}} - (P_0^{31})^2 \frac{\partial \bar{P}_0^{31}}{\partial z} \Big] = 0, \\ \frac{\partial P_1^{31}}{\partial z} + \frac{\partial \bar{P}_1^{31}}{\partial \bar{z}} = 0, \quad P_1^{31} \frac{d\bar{z}}{ds}\Big|_L = g_{t1} - i g_{n1}. \end{gathered} \tag{35}$$

Problem (34) for the zero stress component coincides with the corresponding problem of linear elasticity and has a solution of the form of (30):

$$P_0^{31} = \bar{\varphi}_0'(\bar{z}), \quad \varphi_0(z)\Big|_L = g_0(s) + G_0, \quad g_0(s) = \int_0^s (g_{t0} + i g_{n0})\, ds, \quad G_0 = \text{const}.$$

From the stress P_0^{31}, we find

$$\frac{3}{4b^2} \Big[(\bar{P}_0^{31})^2 \frac{\partial P_0^{31}}{\partial \bar{z}} - (P_0^{31})^2 \frac{\partial \bar{P}_0^{31}}{\partial z} \Big] = \frac{3}{4b^2} \Big[\varphi_0'^2(z) \bar{\varphi}_0''(\bar{z}) - \bar{\varphi}_0'^2(\bar{z}) \varphi_0''(z) \Big] = 2 \frac{\partial f}{\partial z},$$

$$f(z,\bar z)=\frac{3}{8b^2}\Big[\bar\varphi_0''(\bar z)\int\varphi_0'^2(z)\,dz-\bar\varphi_0'^2(\bar z)\varphi_0'(z)\Big],$$

after which problem (35) for the stress P_1^{31} takes the form

$$\frac{\partial P_1^{31}}{\partial z}-\frac{\partial\bar P_1^{31}}{\partial\bar z}+2\,\frac{\partial f}{\partial z}=0,\qquad \frac{\partial P_1^{31}}{\partial z}+\frac{\partial\bar P_1^{31}}{\partial\bar z}=0,\qquad P_1^{31}\,\frac{d\bar z}{ds}\Big|_L=g_{t1}-ig_{n1}.$$

Summation of the equations gives the equation $\partial(P_1^{31}+f)/\partial z=0$, which after integration yields the representation of the stress in terms of the complex potential $\bar\varphi_1'(\bar z)$. Substitution of the representation obtained into the boundary condition yields the boundary-value problem for the potential:

$$P_1^{31}=\bar\varphi_1'(\bar z)-f(z,\bar z),\qquad \varphi_1(z)\Big|_L=g_1(s)+G_1,\qquad g_1=\int\limits_0^s\Big(g_{t1}+ig_{n1}+\bar f\,\frac{dz}{ds}\Big)ds,$$

where $G_1=\mathrm{const}$.

Conformal mapping of the simply connected region S onto a unit circle allows us to represent the stress components in terms of the transformed complex potentials and write the problems for the potentials on the unit circle in a form similar to (31):

$$\begin{gathered}P_0^{31}=\bar\varphi_0'(\bar\zeta)/\bar\omega'(\bar\zeta),\quad \varphi_0(\sigma)=g_0(\sigma),\quad P_1^{31}=\bar\varphi_1'(\bar\zeta)/\bar\omega'(\bar\zeta)-f(\zeta,\bar\zeta),\quad \varphi_1(\sigma)=g_1(\sigma),\\ f(\zeta,\bar\zeta)=\frac{3}{8b^2}\Big[\frac{\bar\Phi_0'(\bar\zeta)}{\bar\omega'(\bar\zeta)}\int\frac{\varphi_0'^2(\zeta)}{\omega'(\zeta)}\,d\zeta-\frac{\bar\varphi_0'^2(\bar\zeta)}{\bar\omega'^2(\bar\zeta)}\,\frac{\varphi_0'(\zeta)}{\omega'(\zeta)}\Big],\quad \Phi_0(\zeta)=\frac{\varphi_0'(\zeta)}{\omega'(\zeta)}.\end{gathered}\tag{36}$$

Thus, in a linear approximation of the small parameter, the stress (32), according to (36), is determined by the potentials and the representation as

$$P^{31}=P_0^{31}+cP_1^{31}=\bar\varphi_0'(\bar\zeta)/\bar\omega'(\bar\zeta)+c(\bar\varphi_1'(\bar\zeta)/\bar\omega'(\bar\zeta)-f(\zeta,\bar\zeta)),$$

where the potentials are found from the corresponding boundary-value problems. As for the dependent stresses (28), their linear approximations in the small parameter are expressed in terms of quantities (36) by the formulas

$$P^{11}=\bar P^{22}=-\frac{(P_0^{31})^2}{2b}+c\,\frac{P_0^{31}}{8b^3}(P_0^{31}(R^2)_0-8b^2P_1^{31}),\qquad P^{32}=\bar P_0^{31}+c\bar P_1^{31},$$

$$P^{12}=-2h+c\,\frac{3(R^2)_0^2-8b^2(R^2)_1}{16b^3},\qquad P^{33}=\frac{(R^2)_0}{4b}-h+c\,\frac{(R^2)_0^2}{32b^3},$$

where the quantities $(R^2)_0$ and $(R^2)_1$ were defined in (33).

Higher-order approximations of stresses can be similarly obtained if we develop quantities (32) as a series in the small parameter.

Let us consider another approximate method of solving problem (24). In this problem, the first equation is satisfied if the stresses (related to U') are expressed in terms of the axial displacement gradients $w(x,y)$:

$$\frac{P_{31}}{U'}=-\frac{\partial w}{\partial x},\qquad \frac{P_{32}}{U'}=-\frac{\partial w}{\partial y}.\tag{37}$$

The second equation is satisfied if the stresses are expressed in terms of the gradient of the stress function $t(x,y)$:

$$P_{31}=\frac{\partial t}{\partial y},\qquad P_{32}=-\frac{\partial t}{\partial x}.\tag{38}$$

Elimination of stresses from equalities (37) and (38) yields the following nonlinear system of equations for the functions t and w, in which U' is defined by relation (11) and the quantity R^2 is defined by formulas (7) and (38):

$$\frac{\partial w}{\partial y}=\frac{1}{U'}\,\frac{\partial t}{\partial x},\quad \frac{\partial w}{\partial x}=-\frac{1}{U'}\,\frac{\partial t}{\partial y},\quad U'=I_+(R^2)+I_-(R^2)-\frac{2b}{3},\quad R^2=\Big(\frac{\partial t}{\partial x}\Big)^2+\Big(\frac{\partial t}{\partial y}\Big)^2.\tag{39}$$

These equations are similar to the following equations of steady-state plane vortex-free flow of an ideal gas with subsonic speed [11]:

$$\frac{\partial\psi}{\partial y}=\frac{\rho}{\rho_0}\,\frac{\partial\varphi}{\partial x},\quad \frac{\partial\psi}{\partial x}=-\frac{\rho}{\rho_0}\,\frac{\partial\varphi}{\partial y},\quad \frac{\rho}{\rho_0}=\Big(1-\frac{v^2}{v_{\max}^2}\Big)^{1/(\chi-1)},\quad v^2=\Big(\frac{\partial\varphi}{\partial x}\Big)^2+\Big(\frac{\partial\varphi}{\partial y}\Big)^2\tag{40}$$

(χ is an adiabatic exponent). The quantities w, t, $1/U'$, and R^2 in Eqs. (39) correspond to the stream function ψ, the velocity potential φ, the relative density ρ/ρ_0, and the square of the gas-flow rate v^2 in Eqs. (40), respectively. However, in contrast to the potentials φ and ψ of gas dynamics, which have no physical meaning, one of the elastic potentials (potential w) has the meaning of axial displacement.

Similarly to (40), Eqs. (39) can be written as a linear system of equations for the same desired functions w and t if the independent variables are appropriately chosen. In (39), nonlinearity is due to the quantity $1/U'$, which depends only on R^2. Therefore, we convert from the Cartesian coordinates x and y in the physical plane to the variables R and J, which are polar coordinates in the plane of stresses P_{31} and P_{32}: $P_{31} = R\cos J$ and $P_{32} = R\sin J$. Using relations (37) and (38), we consider the expression

$$dt + iU'\,dw = (-P_{32}\,dx + P_{31}\,dy) + iU'\Big(-\frac{P_{31}}{U'}\,dx - \frac{P_{32}}{U'}\,dy\Big) = -i(P_{31} - iP_{32})\,dz = -iR\mathrm{e}^{-iJ}\,dz.$$

At $R \neq 0$, it follows that

$$dz = (i\mathrm{e}^{iJ}/R)(dt + iU'\,dw). \tag{41}$$

Assuming that t, w, and z are functions of R and J, from (41) we find

$$\frac{\partial z}{\partial R} = \frac{i\mathrm{e}^{iJ}}{R}\Big(\frac{\partial t}{\partial R} + iU'\,\frac{\partial w}{\partial R}\Big), \qquad \frac{\partial z}{\partial J} = \frac{i\mathrm{e}^{iJ}}{R}\Big(\frac{\partial t}{\partial J} + iU'\,\frac{\partial w}{\partial J}\Big). \tag{42}$$

Equating the mixed derivatives $\partial^2 z/\partial R\partial J$ and $\partial^2 z/\partial J\partial R$ in (42), we obtain

$$\frac{\partial t}{\partial R} + iU'\,\frac{\partial w}{\partial R} = \frac{i}{R}\Big(\frac{\partial t}{\partial J} + iU'\,\frac{\partial w}{\partial J}\Big) + \frac{\partial U'}{\partial R}\,\frac{\partial w}{\partial J}.$$

Separation of the real parts from the imaginary parts in this equality yields the following linear system of equations for t and w:

$$\frac{\partial t}{\partial J} = RU'\,\frac{\partial w}{\partial R}, \qquad \frac{\partial w}{\partial J} = \Big(R\,\frac{d}{dR}\,\frac{U'}{R}\Big)^{-1}\frac{\partial t}{\partial R}. \tag{43}$$

By differentiation, we exclude one of the functions (t or w) from Eqs. (43) and obtain a second-order differential equation for the other function. In particular, the equation for axial displacement has the form

$$\frac{\partial^2 w}{\partial R^2} - \frac{1}{U'}\,\frac{d}{dR}\Big(\frac{U'}{R}\Big)\frac{\partial^2 w}{\partial J^2} + \frac{1}{RU'}\,\frac{d(RU')}{dR}\,\frac{\partial w}{\partial R} = 0.$$

Equations (43) can be further simplified. The coefficients of the derivatives on the right sides of the equalities differ only in sign if the function of $U'(R^2)$ has the form of the radical

$$RU' = -\Big(R\,\frac{d}{dR}\,\frac{U'}{R}\Big)^{-1}, \qquad U' = -(1 + kR^2)^{1/2}, \quad k = \mathrm{const}. \tag{44}$$

For weak nonlinearity [$c = a/(2b) \ll 1$], the dependence $U'(R^2)$ can be written in the form of (44). Indeed, approximation of this dependence by the linear function of the small parameter $U' = m_0(R^2) + cm_1(R^2)$ allows the coefficients to be found from the condition of identical satisfaction of Eq. (9) for quantity U' in this approximation: $m_0^3 + 3cm_0^2m_1 + 2b(m_0^2 + 2cm_0m_1) + 2cbR^2 = 0$. Setting the coefficients at the zeroth and first powers of the parameter equal to zero in this equality, we obtain equations that define the desired quantities as $m_0 = -2b$ and $m_1 = -R^2/(2b)$. Thus, the quantity U' can be written as

$$U' \approx m_0 + cm_1 = -2b(1 + cR^2/(4b^2)) \approx -2b(1 + 2cR^2/(4b^2))^{1/2}. \tag{45}$$

Relations (44) and (45) coincide for $2b = 1$ ($c = a$) and $k = 2c/(4b^2) = 2a$, and dependence (45) has the form

$$U' = -(1 + 2aR^2)^{1/2}, \tag{46}$$

and Eqs. (43) have the form

$$\frac{\partial t}{\partial J} = -R\sqrt{1 + 2aR^2}\,\frac{\partial w}{\partial R}, \quad \frac{\partial w}{\partial J} = R\sqrt{1 + 2aR^2}\,\frac{\partial t}{\partial R}. \tag{47}$$

Converting from the variable R to V and using the relationship between the derivatives with respect to these variables

$$V = \frac{1}{2}\ln\frac{\sqrt{1+2aR^2}+1}{\sqrt{1+2aR^2}-1} \quad \Big(R = \frac{1}{\sqrt{2a}\,\sinh V}\Big), \qquad R\sqrt{1+2aR^2}\,\frac{\partial}{\partial R} = -\frac{\partial}{\partial V}, \tag{48}$$

we write Eqs. (47) for the functions $t(V, J)$ and $w(V, J)$ in the form of the Cauchy–Riemann equations [12]:

$$\frac{\partial t}{\partial J} = \frac{\partial w}{\partial V}, \qquad \frac{\partial w}{\partial J} = -\frac{\partial t}{\partial V}. \tag{49}$$

If we introduce the complex function v of the complex variables Z and $\bar{Z}$:

$$v = w(Z,\bar{Z}) + it(Z,\bar{Z}), \quad Z = J + iV, \quad \bar{Z} = J - iV,$$
$$2\,\frac{\partial}{\partial Z} = \frac{\partial}{\partial J} - i\,\frac{\partial}{\partial V}, \quad 2\,\frac{\partial}{\partial \bar{Z}} = \frac{\partial}{\partial J} + i\,\frac{\partial}{\partial V}, \tag{50}$$

Eqs. (49) can be written in complex form

$$2\,\frac{\partial v}{\partial Z} = \frac{\partial(w+it)}{\partial J} - i\,\frac{\partial(w+it)}{\partial V} = \frac{\partial w}{\partial J} + \frac{\partial t}{\partial V} + i\Big(\frac{\partial t}{\partial J} - \frac{\partial w}{\partial V}\Big) = 0.$$

Integrating the last relation, we find v as an arbitrary function $\bar{W}$ of the variable $\bar{Z}$:

$$v = \bar{W}(\bar{Z}). \tag{51}$$

On the contour L with the equations $x = x(s)$ and $y = y(s)$, according to (24) and (46), the stresses $P_{31} = P_{31}(s)$ and $P_{32} = P_{32}(s)$ and the derivative of the elastic potential $U' = -\sqrt{1 + 2a(P_{31}^2(s) + P_{32}^2(s))}$ are determined. Therefore, Eqs. (37) [compatible by virtue of the first equality in (24)] on L define the displacement

$$w^*(s) = w_0 + \int_0^s \frac{P_{31}x' + P_{32}y'}{\sqrt{1+2a(P_{31}^2+P_{32}^2)}}\,ds, \tag{52}$$

where w_0 is a specified constant. Specified contour stresses determine the quantities $R = \sqrt{P_{31}^2 + P_{32}^2} = R(s)$ and $J = \arctan(P_{32}/P_{31}) = J(s)$ on L, and, in accordance with (48) and (50), they determine the quantities $V = V(s)$ and $J = J(s)$ [$Z = Z(s)$ and $\bar{Z} = \bar{Z}(s)$]. Taking into account the representation of the displacement in terms of the complex potential $w = \operatorname{Re} v = \operatorname{Re} W$ and its value on the contour (52), we obtain the following boundary-value problem for the potential:

$$\operatorname{Re} W(Z)\Big|_L = w^*. \tag{53}$$

The potential $W(Z)$ found from (53) defines the function $w(Z,\bar{Z}) = (W(Z) + \bar{W}(\bar{Z}))/2$, which can be written in terms of the variables z and $\bar{z}$. Integrating equality (41) [after conversion to the variables Z and $\bar{Z}$ with allowance for (46), (48), and (49)], we find the relation

$$z - z_0 = \frac{\sqrt{2a}}{2}\Big[\int e^{iZ} W'(Z)\,dZ + \int e^{i\bar{Z}} \bar{W}'(\bar{Z})\,d\bar{Z}\Big].$$

Adding the complex-conjugate equality to the above relation, we obtain the dependences $z = z(Z,\bar{Z})$ and $\bar{z} = \bar{z}(Z,\bar{Z})$. The Jacobian of this transformation calculated with allowance for relations (46)–(50) is nonzero, i.e.,

$$\frac{\partial(z,\bar{z})}{\partial(Z,\bar{Z})} = \frac{\partial(z,\bar{z})}{\partial(R,J)}\,\frac{\partial(R,J)}{\partial(V,J)}\,\frac{\partial(V,J)}{\partial(Z,\bar{Z})} = \frac{\partial(z,\bar{z})}{\partial(R,J)}\,\frac{R}{2i}\,\sqrt{1+2aR^2} \neq 0,$$

because, according to (42), (43), and (46), we have

$$\frac{\partial(z,\bar{z})}{\partial(R,J)} = \frac{2i}{R^3}\Big[R^2(1+2aR^2)\Big(\frac{\partial w}{\partial R}\Big)^2 + \Big(\frac{\partial w}{\partial J}\Big)^2\Big] \neq 0.$$

Therefore, the transformation is reversible: $Z = Z(z,\bar{z})$ and $\bar{Z} = \bar{Z}(z,\bar{z})$, i.e., there is correspondence between the pairs of variables z, $\bar{z}$ and Z, $\bar{Z}$. By virtue of this correspondence, the function $w(Z,\bar{Z})$ obtained can be written as $w(Z(z,\bar{z}), \bar{Z}(z,\bar{z})) = w(z,\bar{z})$. This function defines the displacement and, according to (37) and the relation $U' = -(1 - 2a|\nabla w|^2)^{-1/2}$, the stresses in the region S.

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